Reparametrization by Arc Length

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For a vector function \( \mathbf{r}(t) \), the input variable \( t \) is referred to as a parameter. The word parameter is also used when referring to the arc length parameter, the context should make it clear which of the two uses is intended. In what follows we will discuss the topic of changing the input parameter for a vector function and what impact this has on the graph of the vector function, the description of the points on the graph and of the speed function.

Recall that the arc length, \( \int_a^b |\mathbf{v}(\tau)| \, d\tau \) is the length of the curve representing \( \mathbf{r}(t) \) from the point corresponding to \( t = a \) to the point corresponding to \( t = b \), more precisely, the length of the curve from the terminal point of the vector \( \mathbf{r}(a) \) to the terminal point of the vector \( \mathbf{r}(b) \). The notation, \( P(a) \), is frequently used for the terminal point of \( \mathbf{r}(a) \). We can then create a function which measures how far we’ve travelled from the terminal point of \( \mathbf{r}(a) \) after a time \( t \), by replacing the upper limit of integration with the variable \( t \). We call this function the arc length parameter, \( s(t) = \int_a^t |\mathbf{v}(\tau)| \, d\tau \) and we refer to \( a \) as the basepoint. Again, more precisely, the basepoint is the terminal point of the vector \( \mathbf{r}(a) \).

1. Let \( \mathbf{r}(t) = (3t, t + 4) \). Find the arc length parameter with basepoint corresponding to \( t = 0 \). Then compute \( s(0) \), \( s(1) \) and \( s(\sqrt{10}) \).

2. Let \( \mathbf{r}(t) = (t, t + 2) \). Find the arc length parameter with basepoint corresponding to \( t = 2 \). Then compute \( s(0) \), \( s(1) \) and \( s(\sqrt{2}) \).

**Note:** For a vector function \( \mathbf{r} \) which is not constant, its arc length parameter satisfies the following

\[
s(t) = \int_{t_0}^t \left| \frac{d\mathbf{r}(\tau)}{d\tau} \right| \, d\tau \quad \Rightarrow \quad \frac{ds}{dt} = \left| \frac{d\mathbf{r}(t)}{dt} \right| > 0
\]

That is, the function \( s(t) \) is increasing. One implication of this is that \( s(t) \) has an inverse function. The following describes the inputs and outputs for \( s(t) \) and \( s^{-1}(t) \)

\[
\begin{align*}
\text{s: } c & \mapsto l \\
\text{s^{-1}: } l & \mapsto c
\end{align*}
\]

That is, the function \( s(t) \) gives the arc length on the curve described by \( \mathbf{r}(t) \) from \( t = a \) (the basepoint) to the point corresponding to \( t = c \). The inverse function takes an arc length \( l \) and gives the value of \( t \) in which the arc length between \( t = a \) and \( t = c \) is \( l \). In particular, if we find the arc length parameter with basepoint \( P(a) \), then \( s(a) = 0 \) and so \( s^{-1}(0) = a \).
3. Let \( \vec{r}(t) = (3t, 2t - 1) \). Find the arc length parameter with basepoint corresponding to \( t = 1 \). Then find the inverse of this function. That is, find \( s^{-1}(t) \).

4. Let \( \vec{r}(t) = (3t, t + 4) \). Using your work from exercise #1, find the inverse of the arc length parameter. That is, find \( s^{-1}(t) \).

5. Let \( \vec{r}(t) = \left( \frac{2}{9} (3t - 1)^2, \frac{2}{3} (t - 1)^2 \right) \). Find the arc length parameter with basepoint corresponding to \( t = 0 \). Then find the inverse of this function. That is, find \( s^{-1}(t) \).

**Example:** Consider the vector functions \( \vec{r}(t) = (2t, t + 1) \) and \( \vec{R}(t) = (6t, 3t + 1) \), which represent lines in the \( xy \)-plane. For \( t = 1 \), the point on the graph for \( \vec{r}(t) \) is \((2, 2)\) and the point on the graph for \( \vec{R}(t) \) is \((6, 4)\). Similarly, the point on the graph for \( \vec{r}(t) \) when \( t = 0 \) is \((0, 1)\) and the point on the graph for \( \vec{R}(t) \) when \( t = 0 \) is also \((0, 1)\). Thus, the graphs pass through the same point. To find the slope of the line represented by \( \vec{r}(t) \), choose two points on the line, such as \((2a, a + 1)\) and \((2b, b + 1)\). The slope through these points is

\[
\frac{\Delta y}{\Delta x} = \frac{(b + 1) - (a + 1)}{2b - 2a} = \frac{b - a}{2(b - a)} = \frac{1}{2}
\]

Similarly, to find the slope of the line represented by \( \vec{R}(t) \), choose two points on the line, such as \((6a, 3a + 1)\) and \((6b, 3b + 1)\). The slope through these points is

\[
\frac{\Delta y}{\Delta x} = \frac{(3b + 1) - (3a + 1)}{6b - 6a} = \frac{3(b - a)}{6(b - a)} = \frac{1}{2}
\]

Therefore, the lines are the same. Notice the relationship, \( \vec{R}(t) = \vec{r}(3t) \).

**Definition:** For a vector function \( \vec{r}(t) \), a reparametrization is another vector function \( \vec{R}(t) \) which has the same graph as \( \vec{r}(t) \), but the points are labelled differently.

In the above example, the point \((4, 3)\) is on the graph of \( \vec{r}(t) \), and corresponds to \( t = 2 \). The point is also on the graph of \( \vec{R}(t) \) and corresponds to \( t = \frac{2}{3} \). So, the points are the same, they are just labelled differently depending on which vector function we are referring to.

**Constructing a Reparametrization:** For a vector function, \( \vec{r}(t) \), a reparametrization will have the form \( \vec{r}(f(t)) \).

Note that there are some restrictions on the function \( f(t) \). In particular, the domain of \( f \) must be \( \mathbb{R} \) and we need the range of \( f \) to match the domain of \( \vec{r} \). Thus, the domain and range of \( f \) must be \( \mathbb{R} \). In addition, we want \( f \) to be monotonic. For example, if \( \vec{r}(t) = (t, t^2) \) and \( f(t) = t^2 \), then \( \vec{r}(f(t)) = (t^2, t^6) \). The graph of \( \vec{r}(f(t)) \) does not match the graph of \( \vec{r}(t) \) (why?).

**Example:** From the previous example, we can see that \( \vec{R}(t) = (6t, 3t + 1) \) is a reparametrization of \( \vec{r}(t) = (2t, t + 1) \) because \( \vec{R}(t) = \vec{r}(f(t)) \) where \( f(t) = 3t \).
**Example:** Let \( \vec{r}(t) = (t^2, e^t) \), \( f(t) = 4t \) and \( g(t) = -2t + 5 \). Then the following are both reparametrizations of \( \vec{r}(t) \).

\[
\vec{r}(f(t)) = ((4t)^2, e^{4t}) = (16t^2, e^{4t})
\]
\[
\vec{r}(g(t)) = ((-2t + 5)^2, e^{-2t+5}) = (4t^2 - 20t + 25, e^{-2t+5})
\]

Thus, a vector function can have infinitely many different reparametrizations.

6. Let \( \vec{r}(t) = \left( t, \frac{1}{2}t^2 \right) \).

   a) Sketch the graph of \( \vec{r}(t) \) by first filling in the following table. Label the point on your graph corresponding to \( t = 0 \) and \( t = 1 \).

   
<table>
<thead>
<tr>
<th>input</th>
<th>output</th>
<th>point on graph</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>−1</td>
<td></td>
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<tr>
<td>2</td>
<td></td>
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<tr>
<td>−2</td>
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</tbody>
</table>

   b) Let \( f(t) = 3t \). Sketch the graph of \( \vec{r}(f(t)) \) by first filling in the following table. Label the point on your graph corresponding to \( t = 0 \) and \( t = 1 \).

   
<table>
<thead>
<tr>
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</tr>
</thead>
<tbody>
<tr>
<td>0</td>
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<td>1</td>
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</tbody>
</table>
7. Let $\vec{r}(t) = (t^2 + t, t + 2)$.

a) Sketch the graph of $\vec{r}(t)$ by first filling in the following table. Label the point on your graph corresponding to $t = 0$ and $t = 1$.

<table>
<thead>
<tr>
<th>input</th>
<th>output</th>
<th>point on graph</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$-1$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td></td>
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<tr>
<td>$-2$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$-0.5$</td>
<td></td>
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</tr>
</tbody>
</table>

b) Let $f(t) = t + 1$. Sketch the graph of $\vec{r}(f(t))$ by first filling in the following table. Label the point on your graph corresponding to $t = 0$ and $t = 1$.

<table>
<thead>
<tr>
<th>input</th>
<th>output</th>
<th>point on graph</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$-1$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$-2$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$-1.5$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

In the above exercises, notice that reparametrizing did not change the graph, it only changed the description of each point, as you illustrated by labelling the points corresponding to $t = 0$ and $t = 1$.

Reparametrization by Arc Length

For a vector function $\vec{r}(t)$ and for a fixed basepoint, $P(a)$ (that is, the point on the graph corresponding to $t = a$), we are going to label a point by its distance along the curve from $P(a)$. 
The above provides a visual for the relationship $s(a) = 0$ and so $s^{-1}(0) = a$, that is the same point is either labelled as 0 or as $a$, depending on the vector function description of the curve.

**Definition:** For a vector function $\vec{r}(t)$, the speed function is $s'(t) = |\vec{v}|$.

**Note:** The unit tangent vector is $\vec{T} = \frac{\vec{v}}{|\vec{v}|}$, so that $\vec{v} = |\vec{v}|\vec{T}$. This can then be written as follows

$$\frac{d\vec{r}}{dt} = s'(t)\vec{T}(t)$$

and shows that we can factor the derivative of a vector function into two components. That is, the derivative is the product of the speed and tangent vector which shows that the derivative has a dynamical component and a geometric component.

**Procedure to find the Reparametrization by Arc Length:**
For the vector function $\vec{r}(t)$ with basepoint corresponding to $t = a$.
1. Find the arc length parameter $s(t)$. That is, $s(t) = \int_a^t |\vec{v}(\tau)| \, d\tau$.
2. Find the inverse function, $s^{-1}(t)$.
3. Simplify the composition, $\vec{R}(t) = \vec{r}(s^{-1}(t))$.

8. Let $\vec{r}(t) = \langle 3t, t + 4 \rangle$. Reparametrize $\vec{r}(t)$ by arc length, using $t = 0$ as the basepoint for $s$. Use your work from exercise #4.

9. Let $\vec{r}(t) = \langle 2t + 1, 3t - 1 \rangle$. Reparametrize $\vec{r}(t)$ by arc length, using $t = -1$ as the basepoint for $s$. 
10. Let \( \vec{r}(t) = \langle 2t, t + 1 \rangle \).
   a) Find the point on the graph of \( \vec{r}(t) \) corresponding to \( t = 1 \).
   b) Find the arc length parameter \( s(t) \), with basepoint corresponding to \( t = 0 \).
   c) Find \( s^{-1}(t) \).
   d) Compute \( s^{-1}(0) \).
   e) Compute the arc length of \( \vec{r}(t) \) from \( t = 0 \) to \( t = 1 \).
   f) Reparametrize \( \vec{r}(t) \) by arc length. That is, find \( \vec{R}(t) = \vec{r}(s^{-1}(t)) \).
   g) Evaluate \( \vec{R}(t) \) using the value of the arc length from part c as the input. That is, the \( t \) value is the arc length of \( \vec{r}(t) \) from \( t = 0 \) to \( t = 1 \).
   h) Find the point on the graph of \( \vec{R}(t) \) corresponding to \( t = \sqrt{5} \).
   i) Find the speed function for \( \vec{r}(t) \) and for \( \vec{R}(t) \).

**Note:** One result of reparametrizing a vector function by arc length, gives a speed function which is identically equal to 1. The graph of the reparametrization is identical to the original graph, the only change is the speed on the curve. Thus, in this case the derivative of the reparametrized vector function is identical to the unit tangent vector.

11. Let \( \vec{r}(t) = \langle t - 2, \sqrt{t^3} \rangle \).
   a) Reparametrize \( \vec{r}(t) \) by arc length, using \( t = -\frac{4}{9} \) as the basepoint for \( s \).
   b) Compute \( s^{-1}(0) \).
   c) Show that the speed function for the reparametrization is 1.

12. Let \( \vec{r}(t) = \langle \frac{2}{t^2+1} - 1, \frac{2t}{t^2+1} \rangle \). Prove that the curve is the unit circle centered on the origin, by reparametrizing \( \vec{r}(t) \) by arc length, using \( t = 0 \) as the basepoint for \( s \).