Using Inverse Functions as an Aid to Compute a Derivative Value

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For the first edition of Continuum, I wrote an article illustrating how to use conjugates to compute the derivative of a radical function, for any index. In what follows, I will present an alternative, algebraic, procedure to compute these same derivatives. This technique will also apply to a wider class of functions, but will prove most useful in the case of radicals. As we will see, the method will require that the point of interest be on the $x$-axis. However, we will then generalize the method by showing that we can shift a function vertically so that the corresponding point is on the $x$-axis, before applying the method.

Consider the limit of a difference quotient definition of the derivative,

$$f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$$

When working with this definition, one of the potential challenges is dealing with the $f(a + h)$ term. In what follows, I will discuss a technique that I discovered in which we will use inverse functions as an aid to compute a derivative value, at a specific point. First, consider the following theorem.

**Theorem 1:** For a differentiable function $k(x)$, in which $k(0) = 0$, and for a function $f(x)$ which is differentiable at $x = a$, then

$$f'(a) = \lim_{h \to 0} \frac{f(a + k(h)) - f(a)}{k(h)}$$

**Proof:** Using L’Hopital’s Rule,

$$\lim_{h \to 0} \frac{f(a + k(h)) - f(a)}{k(h)} = \lim_{h \to 0} \frac{f'(a + k(h))k(h) - 0}{k'(h)} = \lim_{h \to 0} f'(a + k(h))$$

Then, interchanging the limit and the function $f'$ and using the fact that $k(h)$ is continuous and passes through the origin,

$$= f'\left(\lim_{h \to 0} a + k(h)\right) = f'(a + k(0)) = f'(a)$$

The next example will illustrate the theorem.

**Example 1:** Let $f(x) = x^2 + 4x$ and let $k(x) = \sin x$. Compute $f'(2)$.
Solution: Using the formula in the above theorem, we get

\[ f'(2) = \lim_{h \to 0} \frac{f(2 + k(h)) - f(2)}{k(h)} = \lim_{h \to 0} \frac{f(2 + \sin h) - f(2)}{\sin h} = \lim_{h \to 0} \frac{(2 + \sin h)^2 + 4(2 + \sin h) - 12}{\sin h} \]

\[ = \lim_{h \to 0} \frac{4 + 4 \sin h + \sin^2 h + 8 + 4 \sin h - 12}{\sin h} = \lim_{h \to 0} \frac{8 \sin h + \sin^2 h}{\sin h} = \lim_{h \to 0} \frac{8 + \sin h}{\sin h} = 8 \]

Regarding the formula in the theorem, if \( a + k(h) \) happened to be the inverse of \( f(h) \), then the first term in the difference quotient would reduce to \( h \) and so we would get

\[ f'(a) = \lim_{h \to 0} \frac{f(a + k(h)) - f(a)}{k(h)} = \lim_{h \to 0} \frac{f(f^{-1}(h)) - f(a)}{k(h)} = \lim_{h \to 0} \frac{h - f(a)}{k(h)} \]

In order for this limit to exist, we would need \( f(a) \) to be 0. That is, \( x = a \) must be an \( x \)-intercept of the function \( f(x) \).

**Example 2:** Compute \( f'(0) \), where \( f(x) = \sin^{-1} x \).

**Solution:** Using the formula in the theorem, the value of the derivative is

\[ f'(0) = \lim_{h \to 0} \frac{f(0 + k(h)) - f(0)}{k(h)} = \lim_{h \to 0} \frac{f(k(h))}{k(h)} = \lim_{h \to 0} \frac{\sin^{-1}(k(h))}{k(h)} \]

If we choose \( k(h) = \sin h \), then \( k(h) \) is the inverse function of \( f \), is differentiable at 0 and \( k(0) = 0 \). Then,

\[ = \lim_{h \to 0} \frac{\sin^{-1}(\sin h)}{\sin h} = \lim_{h \to 0} \frac{h}{\sin h} = 1 \]

In the remaining examples, we will set up the limit of a difference quotient and then find the necessary expression for \( k(h) \).

**Example 3:** Compute \( f'(0) \), where \( f(x) = 3\sqrt{x} \).

**Solution:**

\[ f'(0) = \lim_{h \to 0} \frac{f(0 + k(h)) - f(0)}{k(h)} = \lim_{h \to 0} \frac{f(k(h))}{k(h)} = \lim_{h \to 0} \frac{3\sqrt{k(h)}}{k(h)} \]

With the goal of having the numerator reduce to \( h \), we will choose \( k(h) = h^3 \). Also, for this choice of \( k(h) \), we see that it has the property of being differentiable and passes through the origin. Then,
\[
\lim_{h \to 0} \frac{3\sqrt[k]{h}}{k(h)} = \lim_{h \to 0} \frac{3\sqrt[h^3]{h}}{h} = \lim_{h \to 0} \frac{h}{h^3} = \lim_{h \to 0} \frac{1}{h^2} = \infty
\]

A Slight Generalization of the Above Technique

If \( f(x) \) is differentiable and invertible but \( x = a \) is not an \( x \)-intercept of \( f(x) \) then define \( g(x) = f(x) - f(a) \). Then, \( g'(a) = f'(a) \) and \( x = a \) is an \( x \)-intercept of \( g(x) \). In other words, if we have every requirement for the above technique except for the fact that \( x = a \) is not an \( x \)-intercept of \( f(x) \), then we can shift the function vertically so that we do get an \( x \)-intercept since vertical shifts do not alter the derivative values at the corresponding points. This follows directly from the fact that the derivative of a constant is 0, so that if \( g(x) = f(x) + c \), then \( g'(a) = f'(a) \).

**Example 4:** Compute \( f'(1) \), where \( f(x) = 3\sqrt{x} \).

**Solution:** Since \( x = 1 \) is not an \( x \)-intercept of \( f(x) \), we will shift the graph down one unit and find the derivative at \( x = 1 \) of the shifted function. That is, let \( g(x) = f(x) - 1 \), so that \( g(1) = 0 \) and also \( g'(1) = f'(1) \).

\[
f'(1) = g'(1) = \lim_{h \to 0} \frac{g(1 + k(h)) - g(1)}{k(h)} = \lim_{h \to 0} \frac{g(1 + k(h))}{k(h)}
\]

We want to choose \( k(h) \) so that \( 1 + k(h) \) is the inverse of \( g \). Equivalently, we want the numerator, \( g(1 + k(h)) \), to reduce to \( h \). That is, we want \( \sqrt[3]{1 + k(h)} - 1 = h \). Solving this equation for \( k(h) \), gives \( k(h) = (h + 1)^{3/2} - 1 \). Notice that \( k(h) \) is differentiable at \( h = 0 \), and \( k(0) = 0 \). Then,

\[
\lim_{h \to 0} \frac{g(1 + k(h))}{k(h)} = \lim_{h \to 0} \frac{h}{(h + 1)^{3/2} - 1} = \lim_{h \to 0} \frac{h}{h^3 + 3h^2 + 3h + 1 - 1}
\]

\[
= \lim_{h \to 0} \frac{h}{h(h^2 + 3h + 3)} = \lim_{h \to 0} \frac{1}{h^2 + 3h + 3} = \frac{1}{3}
\]

That is, \( f'(1) = \frac{1}{3} \), as expected.

In example 4 if the index was larger than 3, say \( n \), then expanding the term \( (h + 1)^n \) would’ve been more work. However, notice that when we evaluated the limit we really only needed the second to last term \( (why?) \). The following will prove to be very useful in the case of radicals with a larger index.

**Note:** In the remaining examples, we are going to use the binomial theorem, in the following way,

\[(h + a)^n = hp(h) + a^n\]
That is, expanding \((h + a)^n\) yields a sum of terms in which all but the last have a factor of \(h\). We can factor out the \(h\) from the first terms, leaving a polynomial \(p(h)\), and the last term is \(a^n\). The details for this are as follows,

\[
(h + a)^n = \sum_{k=0}^{n} \binom{n}{k} h^k a^{n-k} = \sum_{k=1}^{n} \binom{n}{k} h^k a^{n-k} + a^n = h \sum_{k=1}^{n} \binom{n}{k} h^{k-1} a^{n-k} + a^n
\]

Then, define \(p(h) = \sum_{k=1}^{n} \binom{n}{k} h^{k-1} a^{n-k}\). In particular, we can write \(p(h)\) as follows,

\[
p(h) = \sum_{k=1}^{n} \binom{n}{k} h^{k-1} a^{n-k} = \binom{n}{1} h^{1-1} a^{n-1} + \sum_{k=2}^{n} \binom{n}{k} h^{k-1} a^{n-k} = na^{n-1} + \sum_{k=2}^{n} \binom{n}{k} h^{k-1} a^{n-k}
\]

so that

\[
p(0) = na^{n-1}
\]

The function value \(p(0)\) is the coefficient of the second to last term in the polynomial \(p(h)\). In example 4, we could’ve written \((h + 1)^3 = hp(h) + 1^3 = hp(h) + 1\), and then \(p(0) = 3(1)^2 = 3\). This will become more useful in the following examples, as we consider expansions with powers larger than 3.

**Example 5:** Compute \(f'(3)\), where \(f(x) = \sqrt[3]{x}\).

**Solution:** Since \(x = 3\) does not correspond to an \(x\)-intercept of \(f(x)\), we define \(g(x) = f(x) - f(3)\). That is, \(g(x) = \sqrt[3]{x} - \sqrt[3]{3}\). Then,

\[
f'(3) = g'(3) = \lim_{h \to 0} \frac{g(3 + k(h)) - g(3)}{k(h)} = \lim_{h \to 0} \frac{g(3 + k(h))}{k(h)} = \lim_{h \to 0} \frac{\sqrt[3]{3 + k(h)} - \sqrt[3]{3}}{k(h)}
\]

The goal in finding \(k(h)\) is that we want the numerator to reduce to \(h\). Again, this is equivalent to \(3 + k(h) = g^{-1}(h)\), so we solve \(\sqrt[3]{3 + k(h)} - \sqrt[3]{3} = h\), for \(k(h)\). Doing so, gives

\[
k(h) = (h + \sqrt[3]{3})^7 - 3
\]

Substituting this into the above limit gives

\[
\lim_{h \to 0} \frac{\sqrt[3]{3 + k(h)} - \sqrt[3]{3}}{k(h)} = \lim_{h \to 0} \frac{\sqrt[3]{3 + (h + \sqrt[3]{3})^7 - 3} - \sqrt[3]{3}}{(h + \sqrt[3]{3})^7 - 3} = \lim_{h \to 0} \frac{h}{(h + \sqrt[3]{3})^7 - 3}
\]

To evaluate this limit using algebra, we could expand the seventh power, but that is much more work than we need to do. We will use the observation, \((h + a)^n = hp(h) + a^n\). That is,
\[
\lim_{h \to 0} \frac{h}{(h + \sqrt{3})^7 - 3} = \lim_{h \to 0} \frac{h}{hp(h) + (\sqrt{3})^7 - 3}
\]

Reducing this expression,

\[
\lim_{h \to 0} \frac{h}{hp(h) + 3 - 3} = \lim_{h \to 0} \frac{h}{hp(h)} = \lim_{h \to 0} \frac{1}{p(h)} = \frac{1}{p(0)}
\]

In the discussion above, we found the function value \(p(0) = na^{n-1}\), where \(n\) is the index and \(a\) is the value at which we are finding the derivative, so that we get the following final result

\[
f'(3) = \frac{1}{7(\sqrt{3})^6}
\]

\[\Box\]

**Note:** In example 3, \(f(k(h)) = h\), so that \(k(h) = f^{-1}(h)\). In example 4, \(x = 1\) was not an \(x\)-intercept of \(f(x)\), so we shifted the function vertically downwards one unit. Then in that example, we found \(g(1 + k(h)) = h\), so that \(1 + k(h) = g^{-1}(h)\) and so \(k(h) = g^{-1}(h) - 1\). In general, the relationship between the original function \(f(x)\), the value of \(a\) and the function \(k(h)\), is expressed as follows

\[
k(h) = f^{-1}(h + f(a)) - a
\]

In example 5, we found \(k(h) = (h + \sqrt{3})^7 - 3\), which has the same form as the boxed formula stated above. If \(x = a\) corresponds to an \(x\)-intercept of \(f(x)\) at the origin (so that \(a = 0\) and \(f(a) = 0\)), then \(k(h)\) is exactly the inverse of \(f\). If \(x = a\) is an \(x\)-intercept of \(f(x)\) but not at the origin, then \(k(h)\) is a vertical shift of the inverse of \(f\). If \(a = 0\), but \(f(a) \neq 0\), then \(k(h)\) is a horizontal shift of \(f^{-1}\). Of course, \(a\) need not be zero and need not correspond to an \(x\)-intercept. The same expression works for \(k(h)\) in each scenario. In what follows, we won’t use the above expression for \(k(h)\), but will find it as we progress through the examples. The overall goal is to illustrate the algebra that is needed to evaluate these derivative values, not to find simplifying formulas.

**Example 6:** Compute \(f'(11)\), where \(f(x) = \sqrt[19]{x}\).

**Solution:** Since \(x = 11\) does not correspond to an \(x\)-intercept of \(f(x)\), we define \(g(x) = f(x) - f(11)\). That is, \(g(x) = \sqrt[19]{x} - \sqrt[19]{11}\). Then,

\[
f'(11) = g'(11) = \lim_{h \to 0} \frac{g(11 + k(h)) - g(11)}{k(h)} = \lim_{h \to 0} \frac{g(11 + k(h))}{k(h)} = \lim_{h \to 0} \frac{\sqrt[19]{11 + k(h)} - \sqrt[19]{11}}{k(h)}
\]

Similar to the previous example, \(k(h) = (h + \sqrt[19]{11}^{19}) - 11\), so that

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\[
\lim_{h \to 0} \frac{\sqrt[19]{11 + k(h)} - \sqrt[19]{11}}{k(h)} = \lim_{h \to 0} \frac{\sqrt[19]{11 + (h + \sqrt[19]{11})^{19} - 11 - \sqrt[19]{11}}}{(h + \sqrt[19]{11})^{19} - 11} = \lim_{h \to 0} \frac{h}{(h + \sqrt[19]{11})^{19} - 11}
\]

\[= \lim_{h \to 0} \frac{h}{hp(h) + (\sqrt[19]{11})^{19} - 11} = \lim_{h \to 0} \frac{h}{hp(h) + 11 - 11} \]

\[= \lim_{h \to 0} \frac{h}{hp(h)} = \lim_{h \to 0} \frac{1}{p(h)} = \frac{1}{p(0)} = \frac{1}{19(\sqrt[19]{11})^{18}} \]

The method presented above works for functions other than radicals, as illustrated by the next example.

**Example 7:** Compute \(f'(5)\), where \(f(x) = \ln x\).

**Solution:** First, define \(g(x) = f(x) - f(5)\).

\[g'(5) = \lim_{h \to 0} \frac{g(5 + k(h)) - g(5)}{k(h)} = \lim_{h \to 0} \frac{g(5 + k(h))}{k(h)} = \lim_{h \to 0} \frac{\ln(5 + k(h)) - \ln 5}{k(h)} \]

Solving the equation \(\ln(5 + k(h)) - \ln 5 = h\), gives

\[k(h) = e^{h+\ln 5} - 5 \]

then substitution back into the limit,

\[\lim_{h \to 0} \frac{\ln(5 + k(h)) - \ln 5}{k(h)} = \lim_{h \to 0} \frac{\ln(e^{h+\ln 5} - 5) - \ln 5}{e^{h+\ln 5} - 5} = \lim_{h \to 0} \frac{h}{e^{h+\ln 5} - 5} \]

The expression inside the limit can be manipulated as follows

\[= \lim_{h \to 0} \frac{h}{e^h e^{\ln 5} - 5} = \lim_{h \to 0} \frac{h}{5 e^h - 5} = \lim_{h \to 0} \frac{h}{5(e^h - 1)} = \frac{1}{5} \lim_{h \to 0} \frac{1}{e^h - 1} = \frac{1}{5} \lim_{h \to 0} \left( e^\frac{h}{h} - 1 \right) = \frac{1}{5} \]

\[\square \]

The preceding examples can be summarized by the following theorem, which is only stated for completeness, and will not be used in any of the remaining examples.

**Theorem 2:** For a function \(f(x)\) which is differentiable at \(x = a\) and is invertible,
\[ f'(a) = \lim_{h \to 0} \frac{h}{f^{-1}(h + f(a)) - a} \]

**Proof:** The limit has indeterminate form \( \frac{0}{0} \), so we can apply L’Hôpital’s Rule,

\[
\lim_{h \to 0} \frac{h}{f^{-1}(h + f(a)) - a} = \lim_{h \to 0} \frac{\frac{d}{dh}[h]}{\frac{d}{dh}[f^{-1}(h + f(a)) - a]} = \lim_{h \to 0} \frac{1}{(f^{-1})'(h + f(a))}
\]

Then, using the formula for the derivative of an inverse,

\[
= \lim_{h \to 0} \left[ \frac{1}{1} \right] = \frac{1}{1} = f'(a)
\]

Since \( f \) is differentiable, then \( f' \) is continuous and since \( f \) is invertible, we know that \( f^{-1} \) is continuous as well. Thus, we can interchange the operation of taking a limit with these two functions,

\[
\left[ \frac{1}{f'(f^{-1}(h + f(a)))} \right] = \left[ \frac{1}{f'(a)} \right] = f'(a)
\]

\( \square \)

In the remaining examples, we will consider functions built by composition. That is, the “inside” function is not just \( x \).

**Example 8:** Compute \( f'(3) \), where \( f(x) = \sqrt{2x + 1} \).

**Solution:** Since \( x = 3 \) is not an \( x \)-intercept of \( f(x) \), we define \( g(x) = f(x) - f(3) \). Then, the derivative of \( f(x) \) evaluated at \( x = 3 \) is

\[
f'(3) = g'(3) = \lim_{h \to 0} \frac{g(3 + k(h)) - g(3)}{k(h)} = \lim_{h \to 0} \frac{g(3 + k(h))}{k(h)} = \lim_{h \to 0} \frac{\sqrt{2(3 + k(h)) + 1} - \sqrt{7}}{k(h)}
\]

To find the expression for \( k(h) \), set the numerator equal to \( h \) and solve for \( k(h) \).

\[
\frac{\sqrt{2(3 + k(h)) + 1} - \sqrt{7}}{k(h)} = h \quad \Rightarrow \quad 2k(h) + 7 = (h + \sqrt{7})^5 \quad \Rightarrow \quad k(h) = \frac{1}{2} (h + \sqrt{7})^5 - \frac{7}{2}
\]
Substituting this expression into the limit,

\[
\lim_{h \to 0} \sqrt{\frac{5}{2} \left( \frac{1}{2} \left( h + \frac{5}{\sqrt{7}} \right)^5 - \frac{7}{4} \right) + 7 - \frac{5}{\sqrt{7}}} = \lim_{h \to 0} \frac{\sqrt{(h + \frac{5}{\sqrt{7}})^5 - 7} - \frac{5}{\sqrt{7}}}{2 h p(h) + \left( \frac{5}{\sqrt{7}} \right)^5 - 7}
\]

\[
= 2 \lim_{h \to 0} \frac{h}{h p(h) + \left( \frac{5}{\sqrt{7}} \right)^5 - 7}
\]

\[
= 2 \lim_{h \to 0} \frac{1}{p(h)} = \frac{2}{p(0)} = \frac{2}{5 \left( \frac{5}{\sqrt{7}} \right)^4}
\]

\[\Box\]

**Note:** Consider the expression, \(( (h + a)^m + b)^n \), for positive integers \(m\) and \(n\). Using a previously discussed formula, we can write this as follows

\[((h + a)^m + b)^n = (hp(h) + a^m + b)^n\]

Using the binomial theorem, this can then be written as follows

\[(hp(h) + a^m + b)^n = (hp(h) + (a^m + b))^n = \sum_{i=0}^n \binom{n}{i} (hp(h))^i (a^m + b)^{n-i}\]

Then pull off the \(i = 0\) term and factor an \(h\) out of the sum,

\[= \left[ \sum_{i=1}^n \binom{n}{i} h^i p^i(h)(a^m + b)^{n-i} \right] + (a^m + b)^n = h \left[ \sum_{i=1}^n \binom{n}{i} h^{i-1} p^i(h)(a^m + b)^{n-i} \right] + (a^m + b)^n\]

Label the summation in brackets in the latter expression as \(q(h)\). The following will show how to compute the value of \(q(0)\).

\[q(h) = \sum_{i=1}^n \binom{n}{i} h^{i-1} p^i(h)(a^m + b)^{n-i}\]

\[= \binom{n}{1} h^{1-1} p^1(h)(a^m + b)^{n-1} + \sum_{i=2}^n \binom{n}{i} h^{i-1} p^i(h)(a^m + b)^{n-i}\]

\[= np(h)(a^m + b)^{n-1} + h \sum_{i=2}^n \binom{n}{i} h^{i-2} p^i(h)(a^m + b)^{n-i}\]
Thus,

\[ q(0) = np(0)(a^m + b)^{n-1} \]

where

\[ ((h + a)^m + b)^n = hq(h) + (a^m + b)^n \]

**Example 9:** Compute \( f'(1) \), where \( f(x) = \sqrt[3]{x} + 4 \).

**Solution:** Define \( g(x) = f(x) - f(1) \). Then,

\[
 f'(1) = g'(1) = \lim_{h \to 0} \frac{g(1 + k(h)) - g(1)}{k(h)} = \lim_{h \to 0} \frac{g(1 + k(h))}{k(h)} = \lim_{h \to 0} \frac{\sqrt[3]{1 + k(h)} + 4 - \sqrt[3]{5}}{k(h)}
\]

Using the same idea as the previous examples, we solve \( \sqrt[3]{1 + k(h)} + 4 - \sqrt[3]{5} = h \), for \( k(h) \). Then,

\[ k(h) = \left( (h + \sqrt{5})^2 - 4 \right)^3 - 1 \]

Which gives the following

\[
\lim_{h \to 0} \frac{\sqrt[3]{1 + k(h)} + 4 - \sqrt[3]{5}}{k(h)} = \lim_{h \to 0} \frac{h}{\left( (h + \sqrt{5})^2 - 4 \right)^3 - 1}
\]

Notice that the denominator has the same form as the expressions discussed in the note preceding this example. Taking advantage of that

\[
\lim_{h \to 0} \frac{h}{\left( (h + \sqrt{5})^2 - 4 \right)^3 - 1} = \lim_{h \to 0} \frac{h}{hq(h) + \left( (\sqrt{5})^2 + (-4) \right)^3 - 1} = \lim_{h \to 0} \frac{h}{hq(h)} = \frac{1}{q(0)}
\]

Finally, using the previously discovered value of \( q(0) \), we get the final result

\[ f'(1) = \frac{1}{3 \cdot 2(\sqrt{5})^{2-1} \left( (\sqrt{5})^2 + (-4) \right)^{3-1}} = \frac{1}{6\sqrt{5}} \]

\[ \square \]
Example 10: Compute $f'(4)$, where $f(x) = \sqrt[3]{x^2 + 1}$.

Solution: Define $g(x) = f(x) - f(4)$. Then,

$$f'(4) = g'(4) = \lim_{h \to 0} \frac{g(4 + k(h)) - g(4)}{k(h)} = \lim_{h \to 0} \frac{g(4 + k(h))}{k(h)} = \lim_{h \to 0} \frac{3(4 + k(h))^2 + 1 - 3\sqrt{17}}{k(h)}$$

As the reader can verify, we will choose $k(h) = \sqrt{(h + \sqrt{17})^3 - 1 - 4}$, so that the above limit becomes

$$= \lim_{h \to 0} \frac{h}{\sqrt{(h + \sqrt{17})^3 - 1 - 4}}$$

To evaluate this limit, we will use a conjugate,

$$= \lim_{h \to 0} \frac{h}{\sqrt{(h + \sqrt{17})^3 - 1 - 4}} \cdot \frac{\sqrt{(h + \sqrt{17})^3 - 1 + 4}}{\sqrt{(h + \sqrt{17})^3 - 1 + 4}} = \lim_{h \to 0} \frac{h}{(h + \sqrt{17})^3 - 1 + 16}$$

Using the binomial theorem again,

$$= \lim_{h \to 0} \frac{h}{hp(h) + (\sqrt{17})^3 - 17} = \lim_{h \to 0} \frac{h}{hp(h)} = \lim_{h \to 0} \frac{p(h)}{p(h)} = \frac{8}{p(0)}$$

In the above example, $p(0) = na^{n-1} = 3(\sqrt{17})^2$, so that $f'(\sqrt[3]{17}) = \frac{8}{3(\sqrt{17})^2}$.

In example 10, we used the conjugate for a square root function because of the squared term in the radicand. If that exponent was a larger integer, say $n$, then we would end up having to use a conjugate for a radical with index $n$. This method is presented in the first edition of Continuum and the final example will illustrate using the method of this article along with the method from that aforementioned article.

Example 11: Compute $f'(2)$, where $f(x) = \sqrt[7]{x^4 - 5}$.

Solution: Define $g(x) = f(x) - f(2)$. Then,

$$f'(2) = g'(2) = \lim_{h \to 0} \frac{g(2 + k(h)) - g(2)}{k(h)} = \lim_{h \to 0} \frac{g(2 + k(h))}{k(h)}$$
\[ \lim_{h \to 0} \frac{\sqrt[4]{(2 + k(h))^4 - 5} - \sqrt[4]{11}}{k(h)} \]

Solving for \( k(h) \), gives \( k(h) = \sqrt[4]{(h + \sqrt[7]{11})^7 + 5} - 2 \). Then, substituting this back into the above limit gives

\[ \lim_{h \to 0} \frac{\sqrt[4]{(2 + \sqrt[4]{(h + \sqrt[7]{11})^7 + 5} - 2)^4} - 5 - \sqrt[4]{11}}{4 \sqrt[4]{(h + \sqrt[7]{11})^7 + 5} + 2} \]

To evaluate this limit, we need to use a conjugate for a radical of index 4 (see the first edition of Continuum). To find the appropriate conjugate, consider the following factorization

\[ a^4 - b^4 = (a - b)(a + b)(a^2 + b^2) \]

So, if the denominator in the above limit is "\( a - b " \) and if we want to take that fourth root to the fourth power, we will multiply by \( (a + b)(a^2 + b^2) \). That is,

\[ \lim_{h \to 0} \frac{h}{4 \sqrt[4]{(h + \sqrt[7]{11})^7 + 5} + 2} \cdot \left( \frac{\sqrt[4]{(h + \sqrt[7]{11})^7 + 5} + 2}{\sqrt[4]{(h + \sqrt[7]{11})^7 + 5} + 2} \right)^2 \]

\[ = \lim_{h \to 0} \frac{h \left( \sqrt[4]{(h + \sqrt[7]{11})^7 + 5} + 2 \right) \left( \left( \sqrt[4]{(h + \sqrt[7]{11})^7 + 5} \right)^2 + 2^2 \right)}{\left( \sqrt[4]{(h + \sqrt[7]{11})^7 + 5} \right)^4 - 2^4} \]

\[ = \lim_{h \to 0} \frac{h \left( \sqrt[4]{(h + \sqrt[7]{11})^7 + 5} + 2 \right) \left( \left( \sqrt[4]{(h + \sqrt[7]{11})^7 + 5} \right)^2 + 2^2 \right)}{(h + \sqrt[7]{11})^7 + 5 - 16} \]
\[
\lim_{h \to 0} \frac{h \left( \sqrt[4]{hp(h)} + \left( \sqrt[7]{11} \right)^7 + 5 + 2 \right) \left( \left( \sqrt[4]{hp(h)} + \left( \sqrt[7]{11} \right)^7 + 5 \right)^2 + 2^2 \right)}{hp(h) + \left( \sqrt[7]{11} \right)^7 - 11}
\]

\[
= \lim_{h \to 0} \frac{\left( \sqrt[4]{hp(h)} + 16 + 2 \right) \left( \left( \sqrt[4]{hp(h)} + 16 \right)^2 + 4 \right)}{p(h)}
\]

\[
= \left( \sqrt[4]{16} + 2 \right) \left( \left( \sqrt[4]{16} \right)^2 + 4 \right)
\]

\[
= \frac{4 \cdot 8}{7(\sqrt[7]{11})^6}
\]

\[
= \frac{32}{7(\sqrt[7]{11})^6}
\]