Finding The Tangent Line to a Rational Function, using Division

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In this article, I will present a method for finding a tangent line to the graph of a rational function, using polynomial division. As we will see, the method will apply only to a certain sub-collection of rational functions and it will require that the point of tangency be on the y-axis. We will then discuss conditions under which we can generalize the method. That is, we will consider points of tangency which are not on the y-axis, and we will see when we are able to use a horizontal shift so that the corresponding point is on the y-axis. To motivate the method, we will begin with an example.

Example 1: Consider the rational function $f(x) = \frac{x^3 + x^2 - x + 4}{x^2 + 1}$. Using polynomial division to write $f(x)$ as a proper rational function, we get

$$f(x) = x + 1 + \frac{-2x + 3}{x^2 + 1}$$

Now find the tangent line to the graph of $f(x)$ at $x = 0$. The point of tangency is $(0, f(0)) = (0, 4)$ and the derivative of $f(x)$ is

$$f'(x) = \frac{(3x^2 + 2x - 1)(x^2 + 1) - (x^3 + x^2 - x + 4)(2x)}{(x^2 + 1)^2}$$

so that the slope of the tangent line is $m = f'(0) = \frac{-1(1 - 0)}{1} = -1$. The tangent line is

$$y - y_1 = m(x - x_1) \quad \Rightarrow \quad y - 4 = -1(x - 0) \quad \Rightarrow \quad y = -x + 4$$

Notice that if we add the quotient and remainder found from using polynomial division, we get

$$(x + 1) + (-2x + 3) = -x + 4 \quad \square$$

That is, the tangent line is the same as the sum of the quotient and the remainder. Is this a very unique case or is there an underlying reason for this to happen more generally?

Note: Before answering the question in general, let’s consider the following function similar to the above example,

$$h(x) = mx + b + \frac{cx + d}{ax^2 + bx + c}$$

We want to see if this function has the property that the tangent line to the graph of $h(x)$ at $x = 0$ is the sum of $mx + b$ and $cx + d$. We also want to see if there are any requirements on the coefficients for this to happen. The point of tangency for $h(x)$ at $x = 0$ is

$$(0, h(0)) = (0, b + \frac{d}{\gamma})$$
The derivative of $h(x)$ is
\[ h'(x) = m + \frac{c(\alpha x^2 + \beta x + \gamma) - (cx + d)(2\alpha x + \beta)}{(\alpha x^2 + \beta x + \gamma)^2} \]

and the slope of the tangent line is
\[ h'(0) = m + \frac{c\gamma - d\beta}{\gamma^2} \]

The equation of the tangent line is
\[ y - y_1 = m(x - x_1) \quad \Rightarrow \quad y - \left(b + \frac{d}{\gamma}\right) = \left(m + \frac{c\gamma - d\beta}{\gamma^2}\right)(x - 0) \]

which reduces to
\[ y = mx + b + \left(\frac{c\gamma - d\beta}{\gamma^2}\right)x + \frac{d}{\gamma} \]

The first two terms are the quotient and now we will set the last two terms equal to the remainder to find conditions on the coefficients. That is,
\[ \left(\frac{c\gamma - d\beta}{\gamma^2}\right)x + \frac{d}{\gamma} = cx + d \quad \Rightarrow \quad \gamma = 1, \beta = 0 \]

That is, the result holds if the denominator has the form $\alpha x^2 + 1$. The following theorem provides the general answer to the question posed above.

Theorem 1: For a rational function $f(x) = \frac{g(x)}{d(x)}$, the tangent line to $f(x)$ at $x = 0$ will be identical to the sum of the quotient and remainder if and only if $f(x)$ can be written in the following form
\[ f(x) = m_1x + b_1 + \frac{r(x)}{d(x)} \]

where the remainder, $r(x)$, is linear and $d(x)$ is a polynomial of degree at least two, and has the form $d(x) = \alpha_kx^k + \alpha_{k-1}x^{k-1} + \cdots + \alpha_2x^2 + 1$.

Proof: The point of tangency is
\[ (0, f(0)) = \left(0, b + \frac{r(0)}{d(0)}\right) \]

To find the slope of the tangent line, we need to differentiate $f(x)$
\[ f'(x) = m + \frac{r'(x)d(x) - r(x)d'(x)}{(d(x))^2} \]

so that the slope of the tangent line is
\[ m = f'(0) = m + \frac{r'(0)d(0) - r(0)d'(0)}{(d(0))^2} \]

Then, the equation of the tangent line is
\[
y - \left( b + \frac{r(0)}{d(0)} \right) = \left( m + \frac{r'(0)d(0) - r(0)d'(0)}{(d(0))^2} \right) (x - 0) 
\]
\[
y = b + \frac{r(0)}{d(0)} + mx + \left( \frac{r'(0)d(0) - r(0)d'(0)}{(d(0))^2} \right) x 
\]
\[
y = mx + b + \left( \frac{r'(0)d(0) - r(0)d'(0)}{(d(0))^2} \right) x + \frac{r(0)}{d(0)} 
\]

We want to find conditions on \( r(x) \) and \( d(x) \) so that \( \left( \frac{r'(0)d(0) - r(0)d'(0)}{(d(0))^2} \right) x + \frac{r(0)}{d(0)} \) is equal to the remainder. Since the function \( r(x) \) is linear, we can write \( r(x) = m_2 x + b_2 \). For this to happen we need to solve the following system of equations
\[
\begin{align*}
\frac{r'(0)d(0) - r(0)d'(0)}{(d(0))^2} &= m_2 \\
\frac{r(0)}{d(0)} &= b_2 
\end{align*}
\]

From the definition of \( r(x) \), we get \( r(0) = b_2 \), so that the second equation above gives \( d(0) = 1 \). Substituting this into the first equation in the system gives
\[
\frac{r'(0)d(0) - r(0)d'(0)}{(d(0))^2} = m_2 \quad \Rightarrow \quad m_2 - b_2 d'(0) = m_2 
\]

This last equation then says that \( d'(0) = 0 \). Knowing that \( d(x) \) is a polynomial with \( d(0) = 1 \), we know that the constant term must be 1. Knowing that \( d'(0) = 0 \), we know that the coefficient of the first degree term in \( d(x) \) must be 0. That is, the form of \( d(x) \) must be \( a_k x^k + a_{k-1} x^{k-1} + \cdots + a_2 x^2 + 1 \).

Example 2: The tangent line to \( f(x) = 3x + 5 + \frac{2x-1}{x^3+4x^2+1} \) at \( x = 0 \) is \( y = 5x + 4 \). □

Corollary: The following is a direct result of the theorem.
\[
 f(x) = m_1 x + b_1 + \frac{m_2 x + b_2}{a_k x^k + a_{k-1} x^{k-1} + \cdots + a_2 x^2 + 1} \quad \Rightarrow \quad f'(0) = m_1 + m_2 
\]
Example 4: Show that \( x = 0 \) is a critical point of \( f(x) = \frac{x^3 + 1}{x^2 + 2} \), using only algebra.

Solution: So that the function has the form in the corollary, we can factor out the 2 from the denominator,

\[
 f(x) = \frac{x^3 + 1}{x^2 + 2} = \frac{1}{2} \left( \frac{x^3 + 1}{\frac{1}{2}x^2 + 1} \right)
\]

Then, consider the function \( g(x) = \frac{x^3 + 1}{\frac{1}{2}x^2 + 1} \), so that \( f(x) = \frac{1}{2} g(x) \) and therefore \( f'(x) = \frac{1}{2} g'(x) \). Using polynomial division on \( g(x) \) gives

\[
 g(x) = 2x + \frac{-2x + 1}{\frac{1}{2}x^2 + 1}
\]

From the corollary, we get \( g'(0) = 2 - 2 = 0 \) and therefore, \( f'(0) = \frac{1}{2} \cdot 0 = 0 \).

\[\square\]

A Slight Generalization of the Above Technique

Consider the function \( f(x) = (x - 3)^2 \). The point of tangency for the tangent line at \( x = 2 \) is \( (2, f(2)) = (2,1) \) and the slope is \( m = f'(2) = 2(2 - 3) = -2 \). Thus, the tangent line is \( y - 1 = -2(x - 2) \), which is \( y_1(x) = -2x + 5 \). Now, consider shifting the graph of \( f(x) \) so that the point of tangency moves to the \( y \)-axis. To accomplish this, the point \( (2,1) \) shifts two units to the left to the point \( (0,1) \). That is, we consider the function \( g(x) = f(x + 2) \). Now find the tangent line to \( g(x) \) at \( x = 0 \). The function expression for \( g(x) \) is \( g(x) = (x - 1)^2 \) and the point of tangency is \( (0, g(0)) = (0, 1) \) and the slope of the tangent line is \( m = g'(0) = 2(0 - 1) = -2 \) so that the tangent line is \( y - 1 = -2(x - 0) \), giving \( y_2(x) = -2x + 1 \). Finally, consider shifting the tangent line \( y_2(x) \) two units to the right,

\[
 y_2(x - 2) = -2(x - 2) + 1 = -2x + 5
\]

The idea in the above discussion will be stated and as a theorem and proved next. We will then return to the idea of finding tangent lines to rational functions, in particular at locations off of the \( y \)-axis.

Theorem 2 (Horizontal Shift of Tangent Lines): If \( g(x) = f(x + c) \) and \( y_1(x) \) is the tangent line to \( g(x) \) at \( x = 0 \) then \( y_2(x) = y_1(x - c) \) is the tangent line to \( f(x) \) at \( x = c \).

Proof: Suppose \( y_1(x) \) is the tangent line to \( g(x) \) at \( x = 0 \). Then,

\[
 y_1 = m_1(x - x_1) + b_1 \quad \Rightarrow \quad y_1 = g'(0)(x - 0) + g(0)
\]

Since \( g(x) = f(x + c) \), then \( g(0) = f(c) \) and \( g'(x) = f'(x + c) \), so that \( g'(0) = f'(0 + c) + f'(c) \),

\[
 \Rightarrow \quad y_1 = f'(c)x + f(c)
\]
Then, \( y_2(x) = y_1(x - c) = f'(c)(x - c) + f(c) \), which is equivalent to

\[
y_2 - f(c) = f'(c)(x - c)
\]

This latter equation is the tangent line to \( f(x) \) at \( x = c \).

\[\square\]

Example 5: Find the tangent line to \( f(x) = \frac{2x^3 + x - 1}{(x-2)^2 + 1} \) at \( x = 2 \), without computing a derivative.

Solution: Since the point of tangency is not on the \( y \)-axis, we will shift the graph horizontally to the left so that the point of tangency will be on the \( y \)-axis. We will find the tangent line for the shifted function, then perform the opposite shift to the tangent line to obtain the tangent line to the original graph. That is, define \( g(x) = f(x + 2) \). Then, the expression for \( g(x) \) is

\[
g(x) = f(x + 2) = \frac{2(x + 2)^3 + (x + 2) - 1}{((x + 2) - 2)^2 + 1} = \frac{2x^3 + 12x^2 + 25x + 17}{x^2 + 1}
\]

Using long division to write the expression as a proper rational function gives,

\[
g(x) = 2x + 12 + \frac{23x + 5}{x^2 + 1}
\]

Then, the tangent line to \( g(x) \) at \( x = 0 \) is

\[
y_2(x) = (2x + 12) + (23x + 5) = 25x + 17
\]

To obtain the tangent line to the original function \( f(x) \), at \( x = 2 \), we will shift the tangent line \( y_2 \) two units to the right. That is,

\[
y_1(x) = y_2(x - 2) = 25(x - 2) + 17 = 25x - 33
\]

\[\square\]

As the previous example illustrates, we can apply the method of finding a tangent line to a rational function using the results of division, if the rational function has the form \( f(x) = \frac{p(x)}{(x-c)^2+1} \), where the degree of \( p(x) \) is 3. The most general case is the following

\[
f(x) = \frac{p(x)}{d(x-c)}
\]

where, the degree of \( p(x) \) is one more than the degree of \( d(x) \) and \( d(x) \) has the form \( a_kx^k + a_{k-1}x^{k-1} + \cdots + a_2x^2 + 1 \).

In the following theorem, note that the first use of \( x = h \) refers to a specific \( x \)-value whereas the second use refers to a vertical line.

Theorem 3: For a rational function \( f(x) = \frac{p(x)}{d(x)} \) such that \( \deg(p(x)) = 3 \) and \( d(x) \) has no real roots, we can
find the tangent line to \( f(x) \) at \( x = h \), where \( x = h \) is the axis of symmetry of \( d(x) \), using only algebra.

**Proof:** Writing \( d(x) \) in standard form, and using the fact that it has no real roots, we can write the following

\[
f(x) = \frac{p(x)}{d(x)} = \frac{p(x)}{a(x-h)^2 + k} = \frac{1}{k} \left[ \frac{p(x)}{a(x-h)^2 + 1} \right]
\]

Example 6: For the function, \( f(x) = \frac{x^3 + 1}{x^2 + 4x + 5} \), we can write

\[
f(x) = \frac{x^3 + 1}{x^2 + 4x + 5} = \frac{x^3 + 1}{(x + 2)^2 + 1}
\]

so that the tangent line can be found at \( x = -2 \), by first doing a horizontal shift. Similarly, consider the rational function \( g(x) = \frac{x^3 + 1}{x^2 + 4x + 7} \), where we can write

\[
g(x) = \frac{x^3 + 1}{x^2 + 4x + 7} = \frac{x^3 + 1}{(x + 2)^2 + 3} = \frac{1}{3} \left( \frac{x^3 + 1}{1/3(x + 2)^2 + 1} \right)
\]

For this function we can also find the tangent line at \( x = -2 \).

If we have a tangent line to a function \( g(x) \) at \( x = a \) and we know that \( g(x) \) and \( f(x) \) are related by \( g(x) = cf(x) \) then how do we obtain the tangent line to \( f(x) \)? The following theorem will state and prove the fact that if we stretch or compress a function, then we correspondingly stretch, or compress, the tangent lines by the same factor.

**Theorem 4:** If \( g(x) = cf(x) \) and if \( y_t \) is the tangent line to \( f(x) \) at \( x = a \), then \( cy_t \) is the tangent line to \( g(x) \) at \( x = a \).

**Proof:** The tangent line to \( f(a) \) at \( x = a \), is \( y_t = f'(a)(x - a) + f(a) \). The point of tangency on the graph of \( g(x) \) is

\( (a, g(a)) \) and the slope is \( m = g'(a) \), but with the relationship \( g(x) = cf(x) \), we can write this information as \( (a, cf(a)) \) and \( m = cf'(a) \), respectively. Thus, the tangent line to \( g(x) \) at \( x = a \) is

\[
y - cf(a) = cf'(a)(x - a) \quad \Rightarrow \quad y = cf'(a)(x - a) + cf(a) \quad \Rightarrow \quad y = c(f'(a)(x - a) + f(a))
\]

The following example will illustrate Theorem 3 and Theorem 4.

**Example 7:** Let \( f(x) = \frac{x^3 + x^2 + x + 1}{x^2 + 6x + 13} \). Find the tangent line to \( f(x) \) at \( x = h \), where \( x = h \) is the axis of symmetry of the denominator.
Solution: Completing the square on the denominator gives the following for \( f(x) \)

\[ f(x) = \frac{x^3 + x^2 + x + 1}{(x + 3)^2 + 4} \]

From the previous theorem, we see that we can find the tangent line to this function at \( x = -3 \), by first shifting the graph three units to the right. That is, we will define \( g(x) = f(x - 3) \) and find the tangent line to \( g(x) \) at \( x = 0 \).

\[ g(x) = f(x - 3) = \frac{(x - 3)^3 + (x - 3)^2 + (x - 3) + 1}{((x - 3) + 3)^2 + 4} = \frac{x^3 - 8x^2 + 22x - 20}{x^2 + 4} \]

Factoring out the 4 from the denominator puts the function expression in the form we need,

\[ = \frac{1}{4} \left[ \frac{x^3 - 8x^2 + 22x - 20}{x^2 + 1} \right] \]

Define \( h(x) = \frac{x^3 - 8x^2 + 22x - 20}{x^2 + 1} \), so that \( g(x) = \frac{1}{4} h(x) \). Then, using polynomial division on \( h(x) \) gives

\[ h(x) = 4x - 32 + \frac{18x + 12}{4x^2 + 1} \]

Thus, the tangent line to \( h(x) \) at \( x = 0 \) is \( y = (4x - 32) + (18x + 12) = 22x - 20 \). By a previous theorem, the tangent line to \( g(x) \) at \( x = 0 \) is \( y = \frac{1}{4} (22x - 20) \) and finally, by shifting three units to the left, the tangent line to \( f(x) \) at \( x = 3 \) is \( y = \frac{1}{4} (22(x + 3) - 20) = \frac{1}{4} (22x + 66 - 20) = \frac{11}{2} x + \frac{23}{2} \).

\[ \square \]

In all of the preceding discussion, we focused on rational functions and certain values of the first derivative, the following theorem provides a basic result regarding a class of rational functions and a value of the second derivative.

\[ \text{Theorem 5: For a rational function of the form,} \]

\[ f(x) = m_1 x + b_1 + \frac{m_2 x + b_2}{a_k x^k + a_{k-1} x^{k-1} + \cdots + a_3 x^3 + 1} \]

\( x = 0 \) will be a zero of \( f''' \), but is not an inflection point.
Proof: To simplify the notation, we will write \( f(x) = m_1 x + b_1 + \frac{r(x)}{d(x)} \), where \( r(x) \) and \( d(x) \) are the functions stated above. The first derivative of \( f(x) \) is

\[
f'(x) = m_1 + \frac{r'd - rd'}{d^2}
\]

Since \( r(x) \) is a linear function, we know that its derivative is \( m_2 \). Substituting this in and taking the second derivative of \( f(x) \) gives

\[
f''(x) = \frac{(m_2 d' - m_2 d' - r d'')d^2 - (m_2 d - r')d^2}{d^4}
\]

Then, the second derivative evaluated at \( x = 0 \) is

\[
f''(0) = \frac{(m_2 d'(0) - m_2 d'(0) - r(0)d''(0))d^2(0) - (m_2 d(0) - r(0)d'(0))2d(0)d'(0)}{d^4(0)}
\]

We know the function values, \( r(0) = b_2, d(0) = 1, d'(0) = 0 \) and \( d''(0) = 0 \), so that the above reduces to

\[
f''(0) = \frac{-b_2 d''(0)}{d^4(0)} = 0
\]

This shows that \( x = 0 \) is a potential inflection point, but it will depend on the sign of \( f''(x) \) on either side of \( x = 0 \). Since the numerator is a polynomial, we can determine its degree. Also, since the denominator will be strictly positive, the sign is completely determined by the numerator. Consider the following function

\[
y = -rd''d^2 - 2dd'(m_2 d - rd')
\]

Let \( \text{deg}(d(x)) \) denote the degree of the function \( d(x) \). Since \( r(x) \) is linear, \( \text{deg}(r(x)) = 1 \). Then, the degree of the first term is

\[
(\text{deg}(d(x)) - 2) \cdot 2\text{deg}(d(x))
\]

which reduces to

\[
2(\text{deg}(d(x)))^2 - 4\text{deg}(d(x))
\]

That is, the first term has even degree. The degree of the second term is

\[
\text{deg}(d(x)) \cdot (\text{deg}(d(x)) - 1) \cdot \text{deg}(d(x))
\]

which reduces to

\[
(\text{deg}(d(x)))^3 - (\text{deg}(d(x)))^2
\]

which is also even. Thus, the numerator is an even degree polynomial and so does not change sign on either side of its zero.